(#) Provjeriti da li f-ja $z=\varphi(x^2+y^2)$, u kojoj je φ diferencijabilna f-ja, zadovoljava jednakost

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$$

R: Primjetimo da je z složena f-ja dvije promjenjive. Ako sa u označimo x^2+y^2 tj. $u=x^2+y^2$ imamo

$$z=\varphi(u)$$

$$\frac{\partial z}{\partial x}=\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x}=\frac{\partial \varphi}{\partial u}\cdot 2x=\varphi_u'\cdot 2x$$

$$\frac{\partial z}{\partial y}=\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial y}=\frac{\partial \varphi}{\partial u}\cdot 2y=\varphi_u'\cdot 2y$$

$$y\cdot\frac{\partial z}{\partial x}=2xy\,\varphi_u'$$

$$x\cdot\frac{\partial z}{\partial y}=2xy\,\varphi_u'$$

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0$$ vrijedi data jednakost

(#) Ako je $p=u^2\ln v$ pri čemu je $u=\frac{x}{y}$ i $v=3x-2y$, odrediti $\frac{\partial p}{\partial x}$ i provjeriti da li vrijedi $\frac{\partial p}{\partial y} = -\frac{2xu}{vy^2}(v\ln v+y)$.

Rj.

Vidimo da je p složena f-ja dvije promjenjive x i y. Koristićemo sljedeću formulu

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

$$\frac{\partial p}{\partial u} = 2u\ln v, \quad \frac{\partial p}{\partial v} = \frac{u^2}{v}, \quad \frac{\partial v}{\partial x} = 3, \quad \frac{\partial v}{\partial y} = -2$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}, \quad \frac{\partial u}{\partial y} = x(-1)y^{-2} = -\frac{x}{y^2}$$

Prema tome

$u=\frac{x}{y}$

$$\frac{3u^2}{v} = \frac{u}{vy}3yu = \frac{u}{vy}3x$$

$$\frac{\partial p}{\partial x} = \frac{2u\ln v}{y} + \frac{3u^2}{v} = \frac{u}{vy}(2v\ln v + 3x)$$

$$\frac{\partial p}{\partial y} = \frac{-2u\ln v\, x}{y^2} - \frac{2u^2}{v} = -\frac{2xu}{vy^2}(v\ln v + y)$$

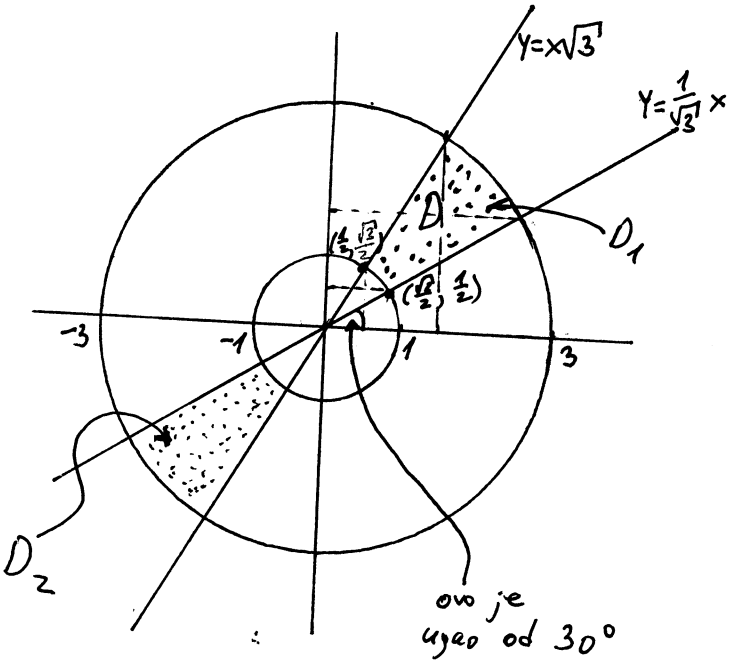
vrijedi data jednakost

$$-\frac{2u^2}{v} = -\frac{2xu}{vy^2}\cdot\underbrace{y^2\cdot u\cdot\frac{1}{x}}_{=y}$$

Izračunati dvojni integral $I=\iint\limits_D \operatorname{arctg}\frac{y}{x}\,dx\,dy$, gdje je

$D=\{(x,y):\ 1\le x^2+y^2\le 9,\ \frac{x}{\sqrt{3}}\le y\le x\sqrt{3}\}$.

Rj: Skicirajmo oblast integracije D.



$\frac{x}{\sqrt{3}}=y \qquad y=x\sqrt{3}$

$y=\frac{1}{\sqrt{3}}x \qquad y=x\sqrt{3}$

$$\begin{cases} x^2+y^2=1\\ y=\frac{1}{\sqrt{3}}x\end{cases}$$

$x^2+\frac{1}{3}x^2=1$

$\frac{4}{3}x^2=1$

$x^2=\frac{3}{4}$

$x_{1,2}=\pm\frac{\sqrt{3}}{2} \Rightarrow y_{1,2}=\pm\frac{1}{2}$

$$\begin{cases} x^2+y^2=9\\ y=\frac{1}{\sqrt{3}}x\end{cases}$$

$x^2+\frac{1}{3}x^2=9$

$\frac{4}{3}x^2=9$

$x^2=\frac{27}{4}$

$x_{3,4}=\pm\frac{3\sqrt{3}}{2}$

$y_{3,4}=\pm\frac{3}{2}$

$$\begin{cases} x^2+y^2=1\\ y=x\sqrt{3}\end{cases}$$

$x^2+3x^2=1$

$4x^2=1$

$x^2=\frac{1}{4}$

$x_{1,2}=\pm\frac{1}{2} \Rightarrow y_{1,2}=\pm\frac{\sqrt{3}}{2}$

$$\begin{cases} x^2+y^2=9\\ y=x\sqrt{3}\end{cases}$$

$4x^2=9$

$x^2=\frac{9}{4}$

$x_{3,4}=\pm\frac{3}{2} \Rightarrow y_{3,4}=\pm\frac{3\sqrt{3}}{2}$

Na osnovu dobijenih presjeka možemo nacrtati date prave

Ako uvedemo polarne koordinate $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $dx\,dy=\rho\,d\rho\,d\varphi$

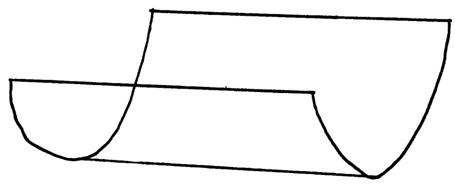
sa slike vidimo da $D \rightsquigarrow D_1\cup D_2$ gdje $D_1:\begin{cases}1\le\rho\le 3\\ \frac{\pi}{6}\le\varphi\le\frac{\pi}{3}\end{cases}$; $D_2:\begin{cases}1\le\rho\le 3\\ \frac{7\pi}{6}\le\varphi\le\frac{4\pi}{3}\end{cases}$

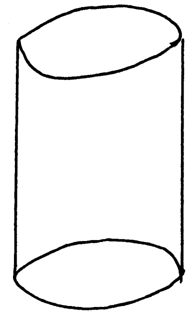
$\frac{y}{x}=\frac{\rho\sin\varphi}{\rho\cos\varphi}=\operatorname{tg}\varphi$

$$I=\iint\limits_D \operatorname{arctg}\frac{y}{x}\,dx\,dy=\left|\begin{matrix}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{matrix}\right|=\iint\limits_{D_1\cup D_2}\varphi\rho\,d\rho\,d\varphi=\ldots=\frac{1}{4}\cdot 8\cdot\frac{\pi^2}{12}+\frac{1}{4}\cdot 8\,\frac{5\pi^2}{12}=\frac{\pi^2}{6}+\frac{5\pi^2}{6}=\pi^2$$

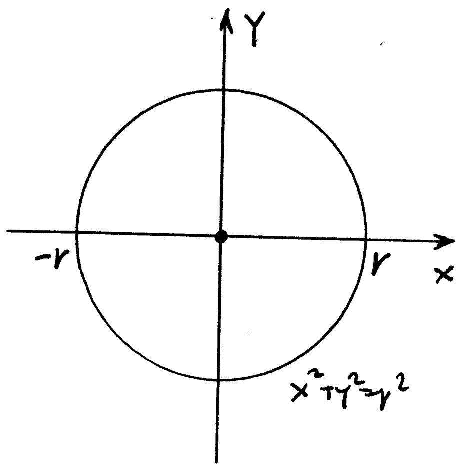
(#) Data je kriva c koja je dobijena kao presjek površina $x^2+y^2=r^2$ i $x^2=rz$ $(r>0)$. Izračunati površinski integral $\iint\limits_S dx\,dy$ gdje je S gornja strana površine koju zatvara kriva c.

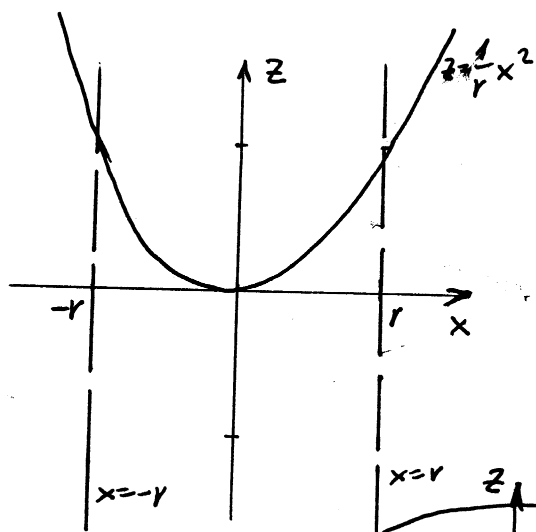
Rj. U prostoru $x^2+y^2=r^2$ predstavlja cilindar, dok $x^2=rz$ predstavlja paraboloid

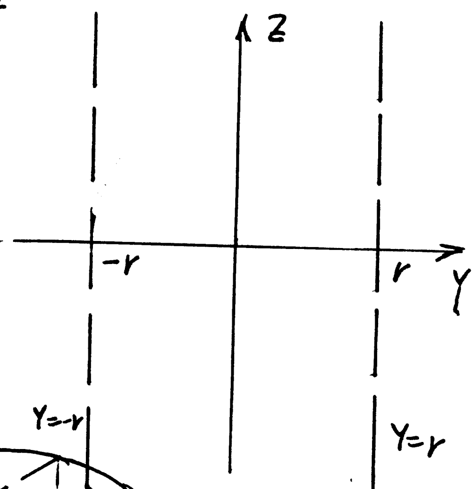




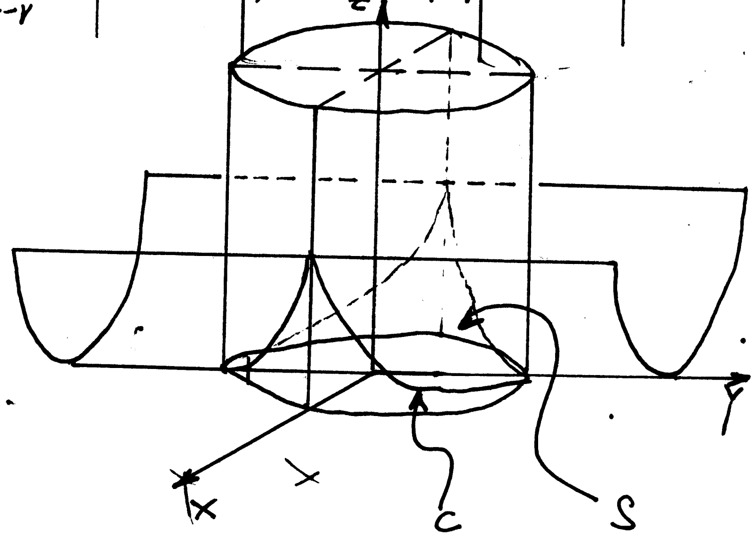
Napravimo presjeke datih površina sa koordinatnim ravnima







Na osnovu presjeka možemo skicirati sliku u prostoru.



$$\iint_S dx\,dy = \left|\begin{array}{l}\bullet \text{ ugao između vektora} \\ \text{normale } \vec{n} \text{ na površ } S \text{ i} \\ z\text{-ose je uvijek između} \\ 0 \text{ i } \frac{\pi}{2} \text{ pa je } \cos\gamma > 0 \\ \bullet \text{ projekcija od } S \text{ na} \\ xOy \text{ ravan je krug} \\ x^2+y^2=r^2\end{array}\right| = +\iint_D dx\,dy =$$

$$= \left|\begin{array}{l} D: x^2+y^2=r^2 \\ \text{uvedimo polarne koordinate} \\ x=\rho\cos\varphi \\ y=\rho\sin\varphi \qquad D \xrightarrow{\text{transformise}} D': \begin{cases} 0\le\rho\le r \\ 0\le\varphi\le 2\pi\end{cases} \\ dx\,dy=\rho\,d\rho\,d\varphi \end{array}\right| = \iint_{D'} \rho\,d\rho\,d\varphi$$

$$= \int_0^r \rho\,d\rho \int_0^{2\pi} d\varphi = \frac{1}{2}\rho^2\Big|_0^r \cdot \varphi\Big|_0^{2\pi} = \pi r^2$$

traženo rješenje

(#) Data su skalarna polja $f = xyz$, $g = xy + yz + zx$

a) Formirati vektorska polja $\vec{a} = \text{grad}\, f$, $\vec{b} = \text{grad}\, g$ i ispitati prirodu vektorskog polja $\vec{a} \times \vec{b}$.

b) Izračunati $\int_C (\vec{a} \times \vec{b}) \cdot d\vec{r}$, gdje je C duž koja spaja tačke $O(0,0,0)$ i $B(1,2,3)$.

R:

a) $$\text{grad}\, f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\left.\begin{aligned} \text{grad}\, f &= (yz, xz, xy) \\ \text{grad}\, g &= (y+z, x+z, x+y) \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{a} &= (yz, xz, xy) \\ \vec{b} &= (y+z, x+z, x+y) \end{aligned}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ yz & xz & xy \\ y+z & x+z & x+y \end{vmatrix} = \left(x^2z + xyz - x^2y - xyz, -(xyz + y^2z - xy^2 - xyz), xyz + yz^2 - xyz - xz^2 \right) =$$

$$= \left(x^2z - x^2y, xy^2 - y^2z, yz^2 - xz^2 \right)$$

Priroda vektorskog polja - misli se da odredimo da li je polje potencijalno ili solenoidno ($\text{rot}(\vec{a} \times \vec{b}) = \vec{0}$ ili $\text{div}(\vec{a} \times \vec{b}) = 0$).

$$\text{rot}\, \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\text{rot}(\vec{a} \times \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2(z-y) & y^2(x-z) & z^2(y-x) \end{vmatrix} = \left(z^2 - y^2, -(z^2 - x^2), y^2 - x^2 \right) \neq \vec{0}$$

$$\underline{\operatorname{div}\vec{v} = \nabla\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}}$$

$$\operatorname{div}(\vec{a}\times\vec{b}) = 2x(z-y) + 2y(x-z) + 2z(y-x) =$$

$$= 2xz - 2xy + 2xy - 2yz + 2yz - 2xz = 0$$

$\vec{a}\times\vec{b}$ je solenoidno polje.

b)

$$\int_C (\vec{a}\times\vec{b})\, d\vec{r} = \left| \begin{array}{l} \vec{a}\times\vec{b} = (x^2(z-y),\ y^2(x-z),\ z^2(y-x)) \\ d\vec{r} = (dx,\ dy,\ dz) \end{array} \right|$$

$$= \int_C x^2(z-y)\,dx + y^2(x-z)\,dy + z^2(y-x)\,dz \overset{(*)}{=}$$

Odredimo parametarski oblik duži OB

$$O(\overset{x_1}{0}, \overset{y_1}{0}, \overset{z_1}{0}) \quad B(\overset{x_2}{1}, \overset{y_2}{2}, \overset{z_2}{3})$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad (=t)$$

$$OB: \begin{cases} x = t \\ y = 2t \\ z = 3t \\ 0 \le t \le 1 \end{cases} \qquad \begin{array}{l} dx = dt \\ dy = 2\,dt \\ dz = 3\,dt \end{array}$$

$$\overset{(*)}{=} \int_0^1 \left(t^2 \cdot t + 4t^2 \cdot (-2t) + 9t^2 \cdot t\right) dt =$$

$$= \int_0^1 (t^3 - 8t^3 + 9t^3)\,dt = \int_0^1 2t^3\,dt = 2\cdot\frac{t^4}{4}\Big|_0^1 = \frac{1}{2} \quad \text{traženo rješenje}$$